

CONVERGENCE RATE AND ACCELERATION OF CLENshaw-CURTIS QUADRATURE FOR FUNCTIONS WITH ENDPOINT SINGULARITIES

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ABSTRACT. In this paper, we study the rate of convergence of Clenshaw-Curtis quadrature for functions with endpoint singularities in X^s , where X^s denotes the space of functions whose Chebyshev coefficients decay asymptotically as $a_k = \mathcal{O}(k^{-s-1})$ for some positive s . For such a subclass of X^s , we show that the convergence rate of $(n+1)$ -point Clenshaw-Curtis quadrature is $\mathcal{O}(n^{-s-2})$. Furthermore, an asymptotic error expansion for Clenshaw-Curtis quadrature is presented which enables us to employ some extrapolation techniques to accelerate its convergence. Numerical examples are provided to confirm our analysis.

1. INTRODUCTION

The evaluation of the definite integral

$$(1.1) \quad I[f] := \int_{-1}^1 f(x) dx,$$

is one of the fundamental and important research topics in the field of numerical analysis [2]. Given a set of distinct nodes $\{x_j\}_{j=0}^n$, an interpolatory quadrature rule of the form

$$(1.2) \quad Q_n[f] := \sum_{j=0}^n w_j f(x_j),$$

can be constructed to approximate the above integral by requiring $I[f] = Q_n[f]$ whenever $f(x)$ is a polynomial of degree n or less. In order to obtain a stable quadrature rule, the quadrature nodes with the Chebyshev density $\mu(x) = 1/\sqrt{1-x^2}$ are preferable. Ideal candidates are the roots or extrema of classical orthogonal polynomials such as Chebyshev and Legendre polynomials.

Clenshaw-Curtis quadrature rule, which is the interpolatory quadrature formula based on the extrema of Chebyshev polynomials, has attracted considerable attention in the past few decades. Let $\{x_j\}_{j=0}^n$ be the Clenshaw-Curtis points or the Chebyshev-Lobatto points

$$(1.3) \quad x_j = \cos\left(\frac{j\pi}{n}\right), \quad j = 0, \dots, n.$$

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Then the Clenshaw-Curtis quadrature rule is

$$(1.4) \quad I_n^C[f] := \sum_{j=0}^n w_j f(x_j),$$

where the quadrature weights are given explicitly by [2, p. 86]

$$(1.5) \quad w_j = \frac{4\delta_j}{n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\delta_{2k}}{1-4k^2} \cos\left(\frac{2jk\pi}{n}\right),$$

and the coefficients δ_j are defined as

$$(1.6) \quad \delta_j = \begin{cases} 1/2, & j = 0 \text{ or } j = n, \\ 1, & \text{otherwise.} \end{cases}$$

Here $\lfloor \cdot \rfloor$ denotes the integer part. It is well known that the quadrature weights are all positive and can be computed in only $\mathcal{O}(n \log n)$ operations by the inverse Fourier transform [21].

Clenshaw-Curtis quadrature rule with $(n+1)$ -point is exact for polynomials of degree less than or equal to n . However, its performance for differentiable functions is comparable with the classic Gauss-Legendre quadrature which is exact for polynomials of degree up to $2n+1$. This remarkable accuracy makes it extraordinarily attractive and many studies have been done on the error behaviour of the Clenshaw-Curtis quadrature (see, for example, [10, 11, 14, 17, 18, 22]). In particular, Trefethen in [17] presented a comprehensive comparison of error bounds of Gauss and Clenshaw-Curtis quadrature rules for analytic and differentiable functions. For the latter, an $\mathcal{O}(n^{-s})$ bound was established for functions belong to X^s , where X^s denotes the space of functions whose Chebyshev coefficients decay asymptotically as $a_k = \mathcal{O}(k^{-s-1})$ for some positive s . More recently, Xiang and Bornemann in [22] presented a more accurate estimate and showed that the optimal rate of convergence of Clenshaw-Curtis quadrature rule for $f \in X^s$ is $\mathcal{O}(n^{-s-1})$.

In this work, we are interested in the rate of convergence of Clenshaw-Curtis quadrature for the integrals $\int_{-1}^1 f(x)dx$, where the integrands $f(x)$ have singularities at one or both endpoints. More specifically, we assume that

$$(1.7) \quad f(x) = (1-x)^\alpha (1+x)^\beta g(x),$$

where $\alpha, \beta \geq 0$ are not integers simultaneously and $g(x) \in C^\infty[-1, 1]$. Note that the assumption $\alpha, \beta \geq 0$ is due to the fact that Clenshaw-Curtis quadrature needs to evaluate the values of the integrand $f(x)$ at both endpoints. When such kind of functions belong to the space X^s where s is determined by the strength of singularities of f , however, we will show that the optimal rate of convergence of Clenshaw-Curtis quadrature for evaluating the integrals $\int_{-1}^1 f(x)dx$ is $\mathcal{O}(n^{-s-2})$, which is one power of n better than that given in [22]. Furthermore, we also extend our analysis to functions with algebraic-logarithmic endpoint singularities of the form

$$(1.8) \quad f(x) = (1-x)^\alpha (1+x)^\beta \log(1-x)g(x),$$

where α is a positive integer and $\beta \geq 0$ and $g(x) \in C^\infty[-1, 1]$. Similarly, we show that the optimal rate of convergence of Clenshaw-Curtis quadrature is also $\mathcal{O}(n^{-s-2})$ if $f(x)$ belongs to X^s .

Apart from the close connection with the FFT, another particularly significant advantage of Clenshaw-Curtis quadrature is that its quadrature nodes are nested. This means that it is possible to accelerate the convergence of Clenshaw-Curtis quadrature by using some extrapolation schemes. In Section 4, we shall explore the asymptotic expansion of the error of Clenshaw-Curtis quadrature for functions with endpoint singularities. An asymptotic series in negative powers of n is derived for even n , which allows to employ some extrapolation schemes, such as the Richardson extrapolation approach, to accelerate the convergence of Clenshaw-Curtis quadrature. Thus, comparing with Gauss-Legendre quadrature, Clenshaw-Curtis quadrature is a more attractive scheme for computing the integrals whose integrands have endpoint singularities.

The rate of convergence of Gauss-Legendre quadrature for functions with endpoint singularities has been investigated considerably in the past decades (see [1, 7, 12, 13, 15, 16, 20] and references therein). For example, for functions like $f(x) = (1-x)^\alpha g(x)$ where $\alpha > -1$ is not an integer and $g(x)$ is sufficiently smooth, Rabinowitz in [12, 13] and Luninsky and Rabinowitz in [7] have shown that the asymptotic error estimate of the n -point Gauss-Legendre quadrature is $\mathcal{O}(n^{-2\alpha-2})$ as $n \rightarrow \infty$. On the other hand, Verlinden in [20] and Sidi in [16] further studied the asymptotic expansion of the error of the Gauss-Legendre quadrature for functions with algebraic and algebraic-logarithmic endpoint singularities. Although the rate of convergence and asymptotic error expansion of Gauss-Legendre quadrature for functions with endpoint singularities have been extensively explored, we are still unable to find the corresponding result for the Clenshaw-Curtis quadrature in the literature. This motivates the author to conduct the current research.

The rest of the paper is organized as follows. In the next section, we shall show that the rate of convergence of Clenshaw-Curtis quadrature can be improved to $\mathcal{O}(n^{-s-2})$ if the Chebyshev coefficients of functions in X^s satisfy a more specific condition; see Theorem 2.2 for details. In Section 3 we discuss the asymptotic behaviour of Chebyshev coefficients of functions with endpoints singularities, including algebraic and algebraic-logarithmic singularities. An asymptotic error expansion for Clenshaw-Curtis quadrature is presented in Section 4. This allows us to use some extrapolation schemes for convergence acceleration. We present some numerical examples in Section 5 and give some concluding remarks in Section 6.

2. CONDITIONS FOR ENHANCED CONVERGENCE RATE

In this section, we establish sufficient conditions under which the rate of convergence of Clenshaw-Curtis quadrature for functions in X^s can be further enhanced. We commence our analysis from a helpful lemma.

Lemma 2.1. *For each $k \geq 1$, we have*

$$(2.1) \quad \sum_{r=1}^n \frac{r^{2k}}{4r^2 - 1} = \frac{1}{4^{k-1}} \frac{n(n+1)}{2(2n+1)} + \sum_{j=1}^{2k-1} \nu_j^k n^{2k-j},$$

where

$$(2.2) \quad \nu_{2j+1}^k = \frac{1}{\Gamma(2k-2j)} \sum_{p=1}^{j+1} \frac{\Gamma(2k-2p+1)}{\Gamma(2j-2p+3)} \frac{B_{2j-2p+2}}{4^p}, \quad 0 \leq j \leq k-2,$$

and

$$(2.3) \quad \nu_{2k-1}^k = \sum_{p=1}^{k-1} \frac{1}{4^p} B_{2k-2p}.$$

Here B_j denotes the j -th Bernoulli number ($B_0 = 1, B_2 = \frac{1}{6}, \dots$). Moreover,

$$(2.4) \quad \nu_{2j}^k = \frac{1}{2^{2j+1}}, \quad 1 \leq j \leq k-1.$$

Proof. Let $H(n, k)$ denote the sum on the left hand side of (2.1). It is easy to derive the following recurrence relation

$$(2.5) \quad 4H(n, j+1) = H(n, j) + \sum_{r=1}^n r^{2j}.$$

Let $S(n, j)$ denote the last sum on the right hand side of the above equation. Multiplying both sides of the above equation by 4^{j-1} and summing over j from 1 to $k-1$, we obtain

$$(2.6) \quad H(n, k) = \frac{1}{4^{k-1}} H(n, 1) + \sum_{j=1}^{k-1} \frac{1}{4^{k-j}} S(n, j),$$

where the sum on the right hand side vanishes when $k = 1$. For $H(n, 1)$, straightforward computation gives

$$(2.7) \quad H(n, 1) = \frac{n(n+1)}{2(2n+1)}.$$

Moreover, using the Faulhaber's formula [6, Corollary 3.4], we have

$$(2.8) \quad S(n, k) = \frac{n^{2k+1}}{2k+1} + \frac{n^{2k}}{2} + \sum_{j=1}^k \frac{\Gamma(2k+1)B_{2j}}{\Gamma(2j+1)\Gamma(2k-2j+2)} n^{2k-2j+1}.$$

Substituting (2.7) and (2.8) into (2.6) gives the desired result. \square

In the following we shall present sufficient conditions for the enhanced rate of convergence of Clenshaw-Curtis quadrature.

Theorem 2.2. *Suppose $f \in X^s$ and if the Chebyshev coefficients of $f(x)$ decay asymptotically as*

$$(2.9) \quad a_m = \frac{c(s)}{m^{s+1}} + \mathcal{O}(m^{-s-2}), \quad m \geq m_0,$$

or

$$(2.10) \quad a_m = (-1)^m \frac{c(s)}{m^{s+1}} + \mathcal{O}(m^{-s-2}), \quad m \geq m_0,$$

where $c(s)$ is independent of m . Then, for $n \geq \max\{m_0, 2\}$, the rate of convergence of Clenshaw-Curtis quadrature rule can be improved to

$$(2.11) \quad E_n^C(f) = \mathcal{O}(n^{-s-2}).$$

Proof. In [22], the authors have presented a simple and elegant proof on the rate of convergence of Clenshaw-Curtis quadrature. For the sake of clarity, we shall briefly describe their idea and then give the key observation that leads to (2.11).

Define

$$(2.12) \quad \Delta(n) = \{m \mid m = 2jn + 2r, \ j \geq 1, \ 1 - n \leq 2r \leq n\}.$$

Note that the Clenshaw-Curtis rule is exact for polynomials of degree n and $E_n^C(f) = 0$ for odd functions f . The error of the Clenshaw-Curtis quadrature rule can be written as

$$(2.13) \quad E_n^C(f) = \sum_{m \in \Delta(n)} a_m E_n^C(T_m),$$

where $T_j(x)$ denotes the Chebyshev polynomial of degree j . Moreover, using the aliasing condition, we have that

$$(2.14) \quad E_n^C(T_m) = \frac{2}{1-m^2} - \frac{2}{1-4r^2}, \quad m \in \Delta(n).$$

Substituting this into the reminder $E_n^C(f)$ yields

$$E_n^C(f) = S_1 + S_2,$$

where

$$(2.15) \quad S_1 = \sum_{m \in \Delta(n)} \frac{2a_m}{1-m^2}, \quad S_2 = \sum_{m \in \Delta(n)} \frac{2a_m}{4r^2-1}.$$

From the assumption that $f \in X^s$, it is easy to deduce that $S_1 = \mathcal{O}(n^{-s-2})$. The remaining task is to give an accurate estimate of S_2 . Using the following identities

$$(2.16) \quad \sum_{r=-\infty}^{\infty} \frac{1}{|4r^2-1|} = 2, \quad \sum_{j=1}^{\infty} \frac{1}{j^{s+1}} = \zeta(s+1),$$

where $\zeta(n)$ is the Riemann zeta function, Xiang and Bornemann in [22] deduced that

$$(2.17) \quad \begin{aligned} |S_2| &\leq \sum_{m \in \Delta(n)} \frac{2|a_m|}{|4r^2-1|} \\ &= \sum_{j=1}^{\infty} \sum_{1-n \leq 2r \leq n} \frac{2|a_{2jn+2r}|}{|4r^2-1|} = \mathcal{O}(n^{-s-1}). \end{aligned}$$

Hence, they proved that the convergence rate of the Clenshaw-Curtis quadrature for $f \in X^s$ is $\mathcal{O}(n^{-s-1})$.

In the following, we shall show that if $f \in X^s$ and (2.9) or (2.10) is satisfied, the rate of convergence of the Clenshaw-Curtis quadrature can be further improved. The key observation is that the estimate of S_2 can be further improved to $\mathcal{O}(n^{-s-2})$. Here we only discuss the case (2.9) and the case (2.10) can be analyzed similarly.

For $n \geq \max\{m_0, 2\}$, substituting the asymptotic of a_m into S_2 , we have

$$(2.18) \quad \begin{aligned} S_2 &= \sum_{m \in \Delta(n)} \frac{2a_m}{4r^2-1} \\ &= \sum_{m \in \Delta(n)} \frac{2c(s)}{(4r^2-1)m^{s+1}} + \sum_{m \in \Delta(n)} \frac{2}{(4r^2-1)} \mathcal{O}(m^{-s-2}). \end{aligned}$$

In analogy to the estimate of (2.17), it is easy to deduce that the last sum in the above equation is $\mathcal{O}(n^{-s-2})$, and thus we get

$$\begin{aligned}
 S_2 &= \sum_{m \in \Delta(n)} \frac{2c(s)}{(4r^2 - 1)m^{s+1}} + \mathcal{O}(n^{-s-2}) \\
 &= \sum_{j=1}^{\infty} \sum_{1-n \leq 2r \leq n} \frac{2c(s)}{(4r^2 - 1)(2jn + 2r)^{s+1}} + \mathcal{O}(n^{-s-2}) \\
 (2.19) \quad &= \frac{2c(s)}{(2n)^{s+1}} \sum_{j=1}^{\infty} \frac{1}{j^{s+1}} \sum_{1-n \leq 2r \leq n} \frac{1}{4r^2 - 1} \left(1 + \frac{r}{jn}\right)^{-s-1} + \mathcal{O}(n^{-s-2}).
 \end{aligned}$$

We now consider the asymptotic behaviour of the double sum in the above equation. First, we consider the case that n is even. Rearranging the inner sum, we obtain

$$\begin{aligned}
 &\sum_{1-n \leq 2r \leq n} \frac{1}{4r^2 - 1} \left(1 + \frac{r}{jn}\right)^{-s-1} \\
 (2.20) \quad &= -1 + \sum_{k=1}^{n/2} \frac{1}{4k^2 - 1} \left[\left(1 + \frac{k}{jn}\right)^{-s-1} + \left(1 - \frac{k}{jn}\right)^{-s-1} \right] \\
 &\quad - \frac{1}{n^2 - 1} \left(1 - \frac{1}{2j}\right)^{-s-1}.
 \end{aligned}$$

Utilizing the following binomial series expansion

$$(2.21) \quad (1+x)^{-\beta} = \sum_{k=0}^{\infty} (-1)^k \frac{(\beta)_k}{k!} x^k, \quad |x| < 1,$$

where $(z)_n$ is the Pochhammer symbol, we further get

$$\begin{aligned}
 \sum_{1-n \leq 2r \leq n} \frac{1}{4r^2 - 1} \left(1 + \frac{r}{jn}\right)^{-s-1} &= -1 + \sum_{k=1}^{n/2} \frac{2}{4k^2 - 1} \sum_{q=0}^{\infty} \frac{(s+1)_{2q}}{(2q)!} \left(\frac{k}{jn}\right)^{2q} \\
 (2.22) \quad &\quad - \frac{1}{n^2 - 1} \left(1 - \frac{1}{2j}\right)^{-s-1}.
 \end{aligned}$$

This together with the following identities

$$(2.23) \quad \sum_{k=1}^{n/2} \frac{2}{4k^2 - 1} = \frac{n}{n+1}, \quad \sum_{j=1}^{\infty} \left(j - \frac{1}{2}\right)^{-s-1} = (2^{s+1} - 1)\zeta(s+1),$$

gives

$$\begin{aligned}
 S_2 &= \frac{2c(s)}{(2n)^{s+1}} \left(- \left(\frac{2^{s+1} - 1}{n^2 - 1} + \frac{1}{n+1} \right) \zeta(s+1) \right. \\
 (2.24) \quad &\quad \left. + \sum_{j=1}^{\infty} \frac{1}{j^{s+1}} \sum_{k=1}^{n/2} \frac{2}{4k^2 - 1} \sum_{q=1}^{\infty} \frac{(s+1)_{2q}}{(2q)!} \left(\frac{k}{jn}\right)^{2q} \right) + \mathcal{O}(n^{-s-2}).
 \end{aligned}$$

Next, we explore the asymptotic behaviour of the last term inside the bracket. By the results of Lemma 2.1, we have

$$\begin{aligned}
& \sum_{j=1}^{\infty} \frac{1}{j^{s+1}} \sum_{k=1}^{n/2} \frac{2}{4k^2-1} \sum_{q=1}^{\infty} \frac{(s+1)_{2q}}{(2q)!} \left(\frac{k}{jn} \right)^{2q} \\
&= 2 \sum_{j=1}^{\infty} \frac{1}{j^{s+1}} \sum_{q=1}^{\infty} \frac{(s+1)_{2q}}{(2q)!(jn)^{2q}} \left(\frac{n(n+2)}{4q2(n+1)} + \sum_{k=1}^{2q-1} \nu_k^q \left(\frac{n}{2} \right)^{2q-k} \right) \\
&= \frac{n(n+2)}{n+1} \sum_{j=1}^{\infty} \frac{1}{j^{s+1}} \sum_{q=1}^{\infty} \frac{(s+1)_{2q}}{(2q)!(2jn)^{2q}} \\
&+ 2 \sum_{j=1}^{\infty} \frac{1}{j^{s+1}} \sum_{q=1}^{\infty} \frac{(s+1)_{2q}}{(2q)!(jn)^{2q}} \left(\sum_{k=1}^{q-1} \nu_{2k}^q \left(\frac{n}{2} \right)^{2q-2k} + \sum_{k=1}^q \nu_{2k-1}^q \left(\frac{n}{2} \right)^{2q-2k+1} \right).
\end{aligned}$$

Now using the explicit expression of ν_k^q and after some elementary computations, we arrive at

$$\begin{aligned}
& \sum_{j=1}^{\infty} \frac{1}{j^{s+1}} \sum_{k=1}^{n/2} \frac{2}{4k^2-1} \sum_{q=1}^{\infty} \frac{(s+1)_{2q}}{(2q)!} \left(\frac{k}{jn} \right)^{2q} \\
&= \frac{1}{n^2-1} \sum_{j=1}^{\infty} \frac{1}{j^{s+1}} \sum_{q=1}^{\infty} \frac{(s+1)_{2q}}{(2q)!(2j)^{2q}} \\
&+ \left(\frac{n(n+2)}{n+1} - \frac{n^2}{n^2-1} \right) \sum_{k=1}^{\infty} \frac{(s+1)_{2k} \zeta(s+2k+1)}{(2k)!(2n)^{2k}} \\
&+ \sum_{k=0}^{\infty} \frac{1}{n^{2k+1}} \sum_{j=1}^{\infty} \frac{1}{j^{s+1}} \sum_{\ell=0}^{\infty} \frac{\nu_{2k+1}^{k+\ell+1} (s+1)_{2\ell+2k+2}}{2^{2\ell} (2\ell+2k+2)! j^{2\ell+2k+2}} \\
&= \mathcal{O}(n^{-1}).
\end{aligned} \tag{2.25}$$

Hence, we immediately deduce that

$$S_2 = \mathcal{O}(n^{-s-2}), \quad n \rightarrow \infty. \tag{2.26}$$

Thus, the desired result follows. For the case that n is odd, similar to (2.20), rearranging the summation yields

$$\begin{aligned}
& \sum_{1-n \leq 2r \leq n} \frac{1}{4r^2-1} \left(1 + \frac{r}{jn} \right)^{-s-1} \\
&= -1 + \sum_{k=1}^{\frac{n-1}{2}} \frac{1}{4k^2-1} \left[\left(1 + \frac{k}{jn} \right)^{-s-1} + \left(1 - \frac{k}{jn} \right)^{-s-1} \right].
\end{aligned} \tag{2.27}$$

The remaining argument can be proceeded similarly as the case n is even and we omit the details. This proves the theorem. \square

Remark 2.3. Functions satisfy (2.9) or (2.10) are only a subclass of X^s . We will show in the next section that typical examples are functions with endpoint singularities.

Remark 2.4. If additional terms like

$$(2.28) \quad b_{m,s} = \pm \frac{d(s)}{m^{s+1+\mu}}, \quad \text{or} \quad \pm (-1)^m \frac{d(s)}{m^{s+1+\mu}},$$

where $d(s)$ is independent of m and $0 < \mu < 1$, are added in (2.9) or (2.10). Then, similar to the estimate of S_2 , we can deduce that

$$(2.29) \quad \sum_{m \in \Delta(n)} \frac{2b_{m,s}}{4r^2 - 1} = \mathcal{O}(n^{-s-2-\mu}).$$

Hence, the rate of convergence of Clenshaw-Curtis quadrature rule is also $E_n^C(f) = \mathcal{O}(n^{-s-2})$.

3. ASYMPTOTICS OF CHEBYSHEV COEFFICIENTS OF FUNCTIONS WITH ENDPOINTS SINGULARITIES

We have showed that the rate of convergence of Clenshaw-Curtis quadrature can be improved if the Chebyshev coefficients of $f(x)$ satisfy (2.9) or (2.10). It is natural to raise the following question: what kind of functions satisfy these conditions? In this section we shall give some typical examples, including functions with algebraic and algebraic-logarithmic singularities. Moreover, for each class of functions, we also establish the corresponding rate of convergence of Clenshaw-Curtis quadrature.

3.1. Functions with algebraic singularities. Elliott in [5] and Tuan and Elliott in [19] have investigated the asymptotic of Chebyshev coefficients of the following singular functions

$$(3.1) \quad f(x) = (1 \pm x)^\alpha g(x),$$

where $\alpha > 0$ is not an integer and $g(x)$ is analytic in a region containing the interval $[-1, 1]$. For example, for $f(x) = (1 - x)^\alpha g(x)$, it was shown that its Chebyshev coefficients satisfy [19, Eqn. (4.13)]

$$(3.2) \quad a_n = -\frac{2^{1-\alpha} g(1) \sin(\alpha\pi)}{\pi n^{2\alpha+1}} \Gamma(2\alpha + 1) + \mathcal{O}(n^{-2\alpha-3}).$$

Obviously, functions of this kind satisfy the conditions of Theorem 2.2. For functions with endpoint singularities of the following general form

$$(3.3) \quad f(x) = (1 - x)^\alpha (1 + x)^\beta g(x),$$

where α, β are positive real numbers not integers and $g(x)$ is analytic in a region containing both endpoints, Tuan and Elliott in [19] proposed a complicated technique to separate the singularities with the aid of auxiliary functions and then derived the asymptotic of the Chebyshev coefficients. For more details, we refer the reader to [19].

In the following we shall present a simpler approach to analyze the asymptotic of Chebyshev coefficients of $f(x) = (1 - x)^\alpha (1 + x)^\beta g(x)$ with $\alpha, \beta > -\frac{1}{2}$ are not integers simultaneously. Meanwhile, for the sake of simplicity, we always assume that $g(x) \in C^\infty[-1, 1]$. However, the generalization to the case $g(x) \in C^m[-1, 1]$ for some positive integer m is mathematically straightforward. Note that the assumptions we consider here is more general than that considered in [5, 19].

Before commencing our analysis, we give a useful lemma.

Lemma 3.1. *Suppose that*

$$f(x) = (x - a)^\gamma (b - x)^\delta h(x)$$

with $\gamma, \delta > -1$ and $h(x)$ is m times continuously differentiable for $x \in [a, b]$. Furthermore, define

$$\phi(x) = (x - a)^\gamma h(x), \quad \psi(x) = (b - x)^\delta h(x).$$

Then for large λ ,

$$\begin{aligned} \int_a^b f(x) e^{i\lambda x} dx &\sim e^{i\lambda a} \sum_{k=0}^{m-1} \frac{\psi^{(k)}(a) e^{i\frac{\pi}{2}(k+\gamma+1)} \Gamma(k+\gamma+1)}{\lambda^{k+\gamma+1} k!} \\ &\quad - e^{i\lambda b} \sum_{k=0}^{m-1} \frac{\phi^{(k)}(b) e^{i\frac{\pi}{2}(k-\delta+1)} \Gamma(k+\delta+1)}{\lambda^{k+\delta+1} k!} \\ &\quad + \mathcal{O}(\lambda^{-m-1-\min\{\gamma, \delta\}}), \quad \lambda \rightarrow \infty. \end{aligned} \tag{3.4}$$

Proof. The first proof of this result was given by Erdélyi in [3]. The idea was based on the neutralizer functions together with integration by parts [3, Thm. 3]. If $h(x)$ is analytic in a neighborhood of the interval $[a, b]$, an alternative proof based on the contour integration was given by Lyness [8, Thm. 1.12]. \square

We now give the asymptotic of Chebyshev coefficients of functions with algebraic endpoint singularities.

Theorem 3.2. *For the function $f(x) = (1-x)^\alpha (1+x)^\beta g(x)$ with $\alpha, \beta > -\frac{1}{2}$ are not integers simultaneously and $g(x) \in C^\infty[-1, 1]$, then its Chebyshev coefficients satisfy*

$$\begin{aligned} a_n &\sim \frac{2^{\alpha+\beta+1}}{\pi} \left\{ -\sin(\alpha\pi) \sum_{k=0}^{\infty} \frac{(-1)^k \hat{\psi}^{(2k)}(0) \Gamma(2k+2\alpha+1)}{n^{2k+2\alpha+1} (2k)!} \right. \\ &\quad \left. - (-1)^n \sin(\beta\pi) \sum_{k=0}^{\infty} \frac{(-1)^k \hat{\phi}^{(2k)}(\pi) \Gamma(2k+2\beta+1)}{n^{2k+2\beta+1} (2k)!} \right\}, \quad n \rightarrow \infty, \end{aligned} \tag{3.5}$$

where

$$\hat{\psi}(t) = (\pi - t)^{2\beta} \hat{g}(t), \quad \hat{\phi}(t) = t^{2\alpha} \hat{g}(t), \tag{3.6}$$

and

$$\hat{g}(t) = (t^{-1} \sin(t/2))^{2\alpha} ((\pi - t)^{-1} \cos(t/2))^{2\beta} g(\cos(t)). \tag{3.7}$$

These values $\hat{\psi}^{(2s)}(0)$ and $\hat{\phi}^{(2s)}(\pi)$ can be calculated explicitly using the L'Hôpital's rule. Here we give the first several values

$$\hat{\psi}(0) = \frac{g(1)}{2^{2\alpha}}, \quad \hat{\phi}(\pi) = \frac{g(-1)}{2^{2\beta}}, \tag{3.8}$$

and

$$\hat{\psi}''(0) = -\frac{g(1)}{2^{2\alpha+1}} \left(\frac{\alpha}{3} + \beta \right) - \frac{g'(1)}{2^{2\alpha}}, \quad \hat{\phi}''(\pi) = -\frac{g(-1)}{2^{2\beta+1}} \left(\alpha + \frac{\beta}{3} \right) + \frac{g'(-1)}{2^{2\beta}}. \tag{3.9}$$

Proof. First, make a change of variable $x = \cos t$, we have

$$(3.10) \quad \begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi f(\cos t) \cos(nt) dt \\ &= \frac{2^{\alpha+\beta+1}}{\pi} \int_0^\pi t^{2\alpha} (\pi-t)^{2\beta} \hat{g}(t) \cos(nt) dt, \end{aligned}$$

where $\hat{g}(t)$ is defined as in (3.7). It is easy to see that $\hat{g}(t) \in C^\infty[0, \pi]$. On the other hand, we observe that $\hat{\psi}(t)$ defined in (3.6) is infinitely differentiable at $t = 0$ while $\hat{\phi}(t)$ is infinitely differentiable at $t = \pi$, and

$$\hat{\psi}(-t) = \hat{\psi}(t), \quad \hat{\phi}(\pi+t) = \hat{\phi}(\pi-t).$$

Hence, it holds that

$$(3.11) \quad \hat{\psi}^{(2k+1)}(0) = 0, \quad \hat{\phi}^{(2k+1)}(\pi) = 0, \quad k \geq 0.$$

The desired result then follows from applying Lemma 3.1 with $m = \infty$ to the integral (3.10). \square

Remark 3.3. The assumption $\alpha, \beta > -\frac{1}{2}$ can not be relaxed to $\alpha, \beta > -1$ since the Chebyshev coefficients a_n of the function $f(x) = (1-x)^\alpha(1+x)^\beta g(x)$ will be divergent if one of α and β is less than or equal to $-\frac{1}{2}$.

Corollary 3.4. Under the same assumptions as in Theorem 3.2, if neither α nor β is a nonnegative integer, then the leading term of Chebyshev coefficients of $f(x)$ is given by

$$(3.12) \quad a_n = \begin{cases} (-1)^{n+1} \frac{2^{\alpha-\beta+1} g(-1) \sin(\beta\pi)}{\pi n^{2\beta+1}} \Gamma(2\beta+1) + \mathcal{O}(n^{-\min\{2\alpha+1, 2\beta+3\}}), & \alpha > \beta, \\ -\frac{2 \sin(\alpha\pi) \Gamma(2\alpha+1)}{\pi n^{2\alpha+1}} (g(1) + (-1)^n g(-1)) + \mathcal{O}(n^{-2\alpha-3}), & \alpha = \beta, \\ -\frac{2^{\beta-\alpha+1} g(1) \sin(\alpha\pi)}{\pi n^{2\alpha+1}} \Gamma(2\alpha+1) + \mathcal{O}(n^{-\min\{2\alpha+3, 2\beta+1\}}), & \alpha < \beta. \end{cases}$$

Further, if one of α and β is an integer, then

$$(3.13) \quad a_n = \begin{cases} (-1)^{n+1} \frac{2^{\alpha-\beta+1} g(-1) \sin(\beta\pi)}{\pi n^{2\beta+1}} \Gamma(2\beta+1) + \mathcal{O}(n^{-2\beta-3}), & \text{if } \alpha \text{ is an integer,} \\ -\frac{2^{\beta-\alpha+1} g(1) \sin(\alpha\pi)}{\pi n^{2\alpha+1}} \Gamma(2\alpha+1) + \mathcal{O}(n^{-2\alpha-3}), & \text{if } \beta \text{ is an integer.} \end{cases}$$

Proof. It follows immediately from Theorem 3.2 by taking the leading term of (3.5). \square

Having derived the leading term of the asymptotic of the Chebyshev coefficients, we can define the parameter s such that f belong to the space X^s . Note that our aim is to establish the rate of convergence of Clenshaw-Curtis quadrature for the integral $\int_{-1}^1 f(x) dx$. Thus we restrict our attention to the case $\alpha, \beta \geq 0$.

Definition 3.5. For the function $f(x) = (1-x)^\alpha(1+x)^\beta g(x)$ with $\alpha, \beta \geq 0$ are not integers simultaneously and $g(x) \in C^\infty[-1, 1]$. Define

$$(3.14) \quad s = \begin{cases} 2 \min\{\alpha, \beta\}, & \text{if } \alpha, \beta \text{ are not integers,} \\ 2\alpha, & \text{if } \beta \text{ is an integer,} \\ 2\beta, & \text{if } \alpha \text{ is an integer.} \end{cases}$$

Then, from equations (3.12) and (3.13) we can deduce immediately that $f \in X^s$.

Theorem 3.6. Let $f(x)$ satisfy the assumptions as in Definition 3.5. Then, the rate of convergence of $(n+1)$ -point Clenshaw-Curtis quadrature for the integral $\int_{-1}^1 f(x)dx$ is

$$(3.15) \quad E_n^C[f] = \mathcal{O}(n^{-s-2}),$$

where s is defined as in (3.14).

Proof. If one of α and β is an integer, then we observe from equation (3.13) that the Chebyshev coefficients satisfy the condition of Theorem 2.2, therefore the desired result holds. If neither α nor β is a nonnegative integer, then the desired result holds when $\alpha = \beta$ due to the second equation of (3.12). We now consider the case $\alpha > \beta$: if $\alpha \geq \beta + 1$, then the desired result follows by noting the first equation of (3.12). If $\beta < \alpha < \beta + 1$, using Theorem 3.2 we find that

$$\begin{aligned} a_n = & (-1)^{n+1} \frac{2^{\alpha-\beta+1} g(-1) \sin(\beta\pi)}{\pi n^{2\beta+1}} \Gamma(2\beta+1) \\ & - \frac{2^{\beta-\alpha+1} g(1) \sin(\alpha\pi)}{\pi n^{2\alpha+1}} \Gamma(2\alpha+1) + \mathcal{O}(n^{-2\beta-3}). \end{aligned}$$

This together with Remark 2.4 gives the desired result. Thus, the proof is completed since the argument in the case $\alpha < \beta$ is similar. \square

Corollary 3.7. For functions of the form $f(x) = (1 \pm x)^\alpha g(x)$, where $\alpha > 0$ is not an integer and $g(x) \in C^\infty[-1, 1]$. From Theorem 3.6, we see that the convergence rate of Clenshaw-Curtis quadrature is $E_n^C[f] = \mathcal{O}(n^{-2\alpha-2})$, which is the same as that of Gauss-Legendre quadrature.

Remark 3.8. Not all functions with algebraic endpoint singularities can be expressed in terms of the form $f(x) = (1-x)^\alpha(1+x)^\beta g(x)$. Typical examples are $f(x) = \log(1 + \sin \sqrt{1-x})$ and $f(x) = \arccos(x^{2m})$ where m is a positive integer. The latter function has square root singularities at $x = \pm 1$. However, if we formally define $f(x) = \sqrt{1-x^2} g(x)$ with $g(x) = \arccos(x^{2m})/\sqrt{1-x^2}$. It is easy to verify that $g(x) \in C^\infty[-1, 1]$. Thus, the result of Theorem 3.6 still holds for this function; see Example 5.3 for details.

3.2. Functions with algebraic-logarithmic singularities. In this subsection we consider the asymptotic of the Chebyshev coefficients for functions of the following form

$$(3.16) \quad f(x) = (1-x)^\alpha(1+x)^\beta \log(1-x)g(x),$$

where $\alpha > 0$, $\beta > -\frac{1}{2}$ and $g(x) \in C^\infty[-1, 1]$.

Lemma 3.9. *Suppose that*

$$f(x) = (x-a)^\gamma (b-x)^\delta \log(x-a)h(x)$$

with $\gamma > 0, \delta > -1$ and $h(x)$ is m times continuously differentiable for $x \in [a, b]$. Define

$$\phi(x) = (x-a)^\gamma \log(x-a)h(x), \quad \psi(x) = (b-x)^\delta h(x).$$

Then for large λ ,

$$\begin{aligned} \int_a^b f(x) e^{i\lambda x} dx &= e^{i\lambda a} \sum_{k=0}^{m-1} \frac{\psi^{(k)}(a) e^{i\frac{\pi}{2}(k+\gamma+1)} \Gamma(k+\gamma+1)}{\lambda^{k+\gamma+1} k!} \left(\tilde{\psi}(k+\gamma+1) - \log \lambda + \frac{\pi}{2}i \right) \\ &\quad - e^{i\lambda b} \sum_{k=0}^{m-1} \frac{\phi^{(k)}(b) e^{i\frac{\pi}{2}(k-\delta+1)} \Gamma(k+\delta+1)}{\lambda^{k+\delta+1} k!} \\ &\quad + \mathcal{O}(\lambda^{-m-\gamma-1} \log \lambda) + \mathcal{O}(\lambda^{-m-\delta-1}), \end{aligned}$$

where $\tilde{\psi}(x)$ is the digamma function.

Proof. The idea of Erdélyi's proof can be extended to the current setting in a straightforward way; see [4] for details. If $h(x)$ is analytic, the desired result can be derived by using the technique of contour integration [8, Appendix]. \square

Using the above Lemma, we obtain the following.

Theorem 3.10. *For the function $f(x) = (1-x)^\alpha (1+x)^\beta \log(1-x)g(x)$ with $\alpha > 0$, $\beta > -\frac{1}{2}$ and $g(x) \in C^\infty[-1, 1]$, its Chebyshev coefficients are given asymptotically by*

$$\begin{aligned} a_n &\sim -\frac{2^{\alpha+\beta+1}}{\pi} \sin(\alpha\pi) \sum_{s=0}^{\infty} \frac{(-1)^s \psi_1^{(2s)}(0) \Gamma(2s+2\alpha+1)}{n^{2s+2\alpha+1} (2s)!} \\ &\quad - (-1)^n \frac{2^{\alpha+\beta+1}}{\pi} \sin(\beta\pi) \sum_{s=0}^{\infty} \frac{(-1)^s \phi_1^{(2s)}(\pi) \Gamma(2s+2\beta+1)}{n^{2s+2\beta+1} (2s)!} \\ &\quad - \frac{2^{\alpha+\beta+2}}{\pi} \sum_{s=0}^{\infty} \frac{(-1)^s \hat{\psi}^{(2s)}(0) \Gamma(2s+2\alpha+1)}{n^{2s+2\alpha+1} (2s)!} \left(\sin(\alpha\pi) (\tilde{\psi}(2s+2\alpha+1) - \log n) \right. \\ (3.17) \quad &\quad \left. + \frac{\pi}{2} \cos(\alpha\pi) \right), \quad n \rightarrow \infty, \end{aligned}$$

where

$$(3.18) \quad \psi_1(t) = \hat{\psi}(t) \log(2(t^{-1} \sin(t/2))^2), \quad \phi_1(t) = \hat{\phi}(t) \log(2(\sin(t/2))^2).$$

Proof. The idea is similar to the proof of Theorem 3.2. The change of variable $x = \cos t$ results in

$$\begin{aligned}
 a_n &= \frac{2}{\pi} \int_0^\pi f(\cos t) \cos(nt) dt \\
 &= \frac{2}{\pi} \int_0^\pi (1 - \cos t)^\alpha (1 + \cos t)^\beta \log(1 - \cos t) g(\cos t) \cos(nt) dt \\
 &= \frac{2^{\alpha+\beta+2}}{\pi} \int_0^\pi t^{2\alpha} (\pi - t)^{2\beta} \log(t) \hat{g}(t) \cos(nt) dt \\
 (3.19) \quad &+ \frac{2^{\alpha+\beta+1}}{\pi} \int_0^\pi t^{2\alpha} (\pi - t)^{2\beta} \tilde{g}(t) \cos(nt) dt,
 \end{aligned}$$

where $\hat{g}(t)$ is defined as in (3.7) and

$$(3.20) \quad \tilde{g}(t) = \hat{g}(t) \log(2(t^{-1} \sin(t/2))^2),$$

which is infinitely differentiable on $[0, \pi]$. Note that $\psi_1(t)$ defined in (3.18) is an even function, we have

$$(3.21) \quad \psi_1^{(2k+1)}(0) = 0, \quad k \geq 0.$$

This together with Lemmas 3.9 and 3.1 gives

$$\begin{aligned}
 a_n &\sim \frac{2^{\alpha+\beta+1}}{\pi} \left\{ -\sin(\alpha\pi) \sum_{k=0}^{\infty} \frac{(-1)^k \psi_1^{(2k)}(0) \Gamma(2k+2\alpha+1)}{n^{2k+2\alpha+1} (2k)!} \right. \\
 &\quad \left. - (-1)^n \sum_{k=0}^{\infty} \frac{\phi_2^{(k)}(\pi) \Gamma(k+2\beta+1)}{n^{k+2\beta+1} k!} \sin\left(\beta\pi - \frac{k}{2}\pi\right) \right\} \\
 &\quad + \frac{2^{\alpha+\beta+2}}{\pi} \left\{ -\sum_{k=0}^{\infty} \frac{(-1)^k \hat{\psi}^{(2k)}(0) \Gamma(2k+2\alpha+1)}{n^{2k+2\alpha+1} (2k)!} \left(\sin(\alpha\pi) (\tilde{\psi}(2k+2\alpha+1) - \log n) \right. \right. \\
 (3.22) \quad &\quad \left. \left. + \frac{\pi}{2} \cos(\alpha\pi) \right) - (-1)^n \sum_{k=0}^{\infty} \frac{\phi_3^{(k)}(\pi) \Gamma(k+2\beta+1)}{n^{k+2\beta+1} k!} \sin\left(\beta\pi - \frac{k}{2}\pi\right) \right\},
 \end{aligned}$$

where

$$(3.23) \quad \phi_2(t) = \hat{\phi}(t) \log(2(t^{-1} \sin(t/2))^2), \quad \phi_3(t) = \hat{\phi}(t) \log t.$$

By (3.18), we get $\phi_1(t) = \phi_2(t) + 2\phi_3(t)$. On the other hand, for $k \geq 0$, we have

$$\begin{aligned}
 \phi_1^{(2k+1)}(\pi) &= \left(\hat{\phi}(t) \log(2(\sin(t/2))^2) \right)_{t=\pi}^{(2k+1)} \\
 &= \sum_{j=0}^{2k+1} \binom{2k+1}{j} \hat{\phi}^{(j)}(\pi) (\log(2(\sin(t/2))^2))_{t=\pi}^{(2k+1-j)} \\
 &= 0,
 \end{aligned}$$

where we have used the fact that

$$\hat{\phi}^{(2j+1)}(\pi) = 0, \quad (\log(2(\sin(t/2))^2))^{(2j+1)}(\pi) = 0, \quad j \geq 0.$$

This together with the second and the last sums on the right hand side of (3.22) gives the desired result. This completes the proof. \square

Corollary 3.11. Under the same assumptions as in Theorem 3.10. If α is a positive integer and β is a nonnegative integer, then

$$(3.24) \quad a_n \sim -2^{\alpha+\beta+1} \cos(\alpha\pi) \sum_{k=0}^{\infty} \frac{(-1)^k \hat{\psi}^{(2k)}(0) \Gamma(2k+2\alpha+1)}{n^{2k+2\alpha+1} (2k)!}.$$

If α is a positive integer and β is not a nonnegative integer, then

$$(3.25) \quad a_n = \begin{cases} \frac{(-1)^{n+1} 2^{\alpha-\beta+1} g(-1) \sin(\beta\pi)}{\pi n^{2\beta+1}} \Gamma(2\beta+1) \log 2 + \mathcal{O}(n^{-\min\{2\alpha+1, 2\beta+3\}}), & \alpha > \beta, \\ -\frac{2^{\beta-\alpha+1} g(1) \cos(\alpha\pi)}{n^{2\alpha+1}} \Gamma(2\alpha+1) + \mathcal{O}(n^{-\min\{2\alpha+3, 2\beta+1\}}), & \alpha < \beta. \end{cases}$$

Proof. It follows directly from Theorem 3.10. \square

Again, we define the parameter s such that $f \in X^s$. Meanwhile, we restrict our attention to the case $\alpha > 0$ and $\beta \geq 0$ since our aim is to derive the optimal rate of convergence of Clenshaw-Curtis quadrature for the integral $\int_{-1}^1 f(x) dx$.

Definition 3.12. For the function $f(x) = (1-x)^\alpha (1+x)^\beta \log(1-x) g(x)$ with α a positive integer and $\beta \geq 0$ and $g(x) \in C^\infty[-1, 1]$. Define

$$(3.26) \quad s = \begin{cases} 2\alpha, & \text{if } \beta \text{ is an integer,} \\ 2 \min\{\alpha, \beta\}, & \text{otherwise.} \end{cases}$$

From the above corollary we see that $f \in X^s$.

Theorem 3.13. Let $f(x)$ satisfy the assumptions as in Definition 3.12. Then, the rate of convergence of $(n+1)$ -point Clenshaw-Curtis quadrature for the integral $\int_{-1}^1 f(x) dx$ is

$$(3.27) \quad E_n^C[f] = \mathcal{O}(n^{-s-2}),$$

where s is defined as in (3.26).

Proof. The proof is similar to the proof of Theorem 3.6. \square

Remark 3.14. For the case that α is not a positive integer, from (3.17) we observe that there exists a factor $\log n$ in the third summation. Therefore, it is reasonable to expect that the convergence rate of Clenshaw-Curtis quadrature would be slightly slower than $\mathcal{O}(n^{-s-2})$.

4. EXTRAPOLATION METHODS FOR ACCELERATING CLENSHAW-CURTIS QUADRATURE

Comparing with Gauss quadrature, an essential feature of Clenshaw-Curtis quadrature is that its quadrature nodes are nested. This implies that the previous function values can be stored and reused when the number of quadrature nodes is doubled. Therefore, Clenshaw-Curtis quadrature is a particularly ideal candidate for implementing an automatic quadrature rule in practical computations. In this section, we shall extend our analysis in Section 2 and show that it is possible to accelerate the rate of convergence of Clenshaw-Curtis quadrature for functions with endpoint singularities.

Asymptotic expansion of the error of Gauss-Legendre quadrature for functions with endpoint singularities has been investigated in [16, 20]. For example, when $f(x) = (1-x)^\alpha g(x)$ with $\Re(\alpha) > -1$ and $g(x)$ is analytic in a region containing the interval $[-1, 1]$, Verlinden proved that the error of the n -point Gauss-Legendre quadrature admits the following asymptotic expansion [20, Thm. 1]

$$(4.1) \quad E_n^G[f] \sim \sum_{k=1}^{\infty} c_k h^{k+\alpha}, \quad n \rightarrow \infty,$$

where $h = (n + 1/2)^{-2}$ and c_k are constants independent of n . Furthermore, some extrapolation schemes were proposed to accelerate the convergence of Gauss quadrature. Even Verlinden's results reveal an important connection between Gauss quadrature and extrapolation schemes. However, accelerating the Gauss quadrature is expensive since its quadrature nodes are completely distinct if n is changed.

In the following, we shall show that the error of Clenshaw-Curtis quadrature also admits a similar expansion. This together with the nested property of Clenshaw-Curtis points implies that Clenshaw-Curtis quadrature is more advantageous than its Gauss-Legendre counterpart. Since the Clenshaw-Curtis points are nested when n is doubled, we restrict our attention to the case of even n .

Theorem 4.1. *If the Chebyshev coefficients of $f(x)$ satisfy*

$$(4.2) \quad a_n \sim \sum_{k=0}^{\infty} \frac{\mu_k}{n^{d_k}}, \quad n \rightarrow \infty,$$

or

$$(4.3) \quad a_n \sim (-1)^n \sum_{k=0}^{\infty} \frac{\mu_k}{n^{d_k}}, \quad n \rightarrow \infty,$$

where μ_k are constants independent of n and $0 < d_0 < d_1 < \dots$. Then, for even n , the error of the Clenshaw-Curtis quadrature can be expanded as

$$(4.4) \quad E_n^C[f] \sim \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\varsigma_{j,k}}{n^{d_k+2j+1}},$$

where $\varsigma_{j,k}$ are constants independent of n .

Proof. We only prove the case (4.2) since the case (4.3) can be proved similarly. According to (2.13) and (4.2), we have

$$(4.5) \quad \begin{aligned} E_n^C[f] &= \sum_{m \in \Delta(n)} a_m E_n^C(T_m) \\ &\sim \sum_{m \in \Delta(n)} \left(\sum_{k=0}^{\infty} \frac{\mu_k}{m^{d_k}} \right) E_n^C(T_m) \\ &= \sum_{k=0}^{\infty} \mu_k \sum_{m \in \Delta(n)} \frac{1}{m^{d_k}} E_n^C(T_m). \end{aligned}$$

Moreover, from (2.14) we have

$$(4.6) \quad \sum_{m \in \Delta(n)} \frac{1}{m^{d_k}} E_n^C(T_m) = \sum_{m \in \Delta(n)} \frac{2}{m^{d_k}(1-m^2)} + \sum_{m \in \Delta(n)} \frac{2}{m^{d_k}(4r^2-1)}.$$

In the following we shall analyze the asymptotic of these two sums on the right hand side of the above equation. For the first sum, it is easy to see that

$$\begin{aligned}
 \sum_{m \in \Delta(n)} \frac{2}{m^{d_k}(1-m^2)} &= -2 \sum_{j=0}^{\infty} \sum_{m \in \Delta(n)} \frac{1}{m^{d_k+2j+2}} \\
 (4.7) \quad &= \frac{2}{n^{d_k}(n^2-1)} - \sum_{j=0}^{\infty} \frac{1}{2^{d_k+2j+1}} \zeta\left(d_k+2j+2, \frac{n}{2}\right),
 \end{aligned}$$

where $\zeta(s, a)$ is the Hurwitz zeta function. Recall the asymptotic expansion of $\zeta(s, a)$ [9, p. 25]

$$\zeta(s, a) \sim \frac{1}{(s-1)a^{s-1}} + \frac{1}{2a^s} + \frac{1}{\Gamma(s)} \sum_{\ell=1}^{\infty} \frac{B_{2\ell}}{(2\ell)!} \frac{\Gamma(s+2\ell-1)}{a^{2\ell+s-1}}, \quad a \rightarrow \infty,$$

it follows that

$$(4.8) \quad \sum_{m \in \Delta(n)} \frac{2}{m^{d_k}(1-m^2)} \sim \frac{1}{n^{d_k}(n^2-1)} - \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{4^\ell B_{2\ell} \Gamma(2\ell+2j+d_k+1)}{(2\ell)! \Gamma(d_k+2j+2) n^{2\ell+2j+d_k+1}}.$$

For the second sum, by means of the estimate of S_2 with $c(s) = 1$ and $s+1$ replaced by d_k , we see that

$$\begin{aligned}
 \sum_{m \in \Delta(n)} \frac{2}{m^{d_k}(4r^2-1)} &= \frac{2}{(2n)^{d_k}} \sum_{j=1}^{\infty} \frac{1}{j^{d_k}} \sum_{1-n \leq 2r \leq n} \frac{1}{4r^2-1} \left(1 + \frac{r}{jn}\right)^{-d_k} \\
 &= \frac{2}{(2n)^{d_k}} \left\{ -\left(\frac{2^{d_k}-1}{n^2-1} + \frac{1}{n+1}\right) \zeta(d_k) \right. \\
 &\quad + \frac{1}{n^2-1} \sum_{j=1}^{\infty} \frac{1}{j^{d_k}} \sum_{q=1}^{\infty} \frac{(d_k)_{2q}}{(2q)!(2j)^{2q}} \\
 &\quad + \left(\frac{n(n+2)}{n+1} - \frac{n^2}{n^2-1}\right) \sum_{\ell=1}^{\infty} \frac{(d_k)_{2\ell} \zeta(2\ell+d_k)}{(2\ell)!(2n)^{2\ell}} \\
 &\quad \left. + \sum_{i=0}^{\infty} \frac{1}{n^{2i+1}} \sum_{j=1}^{\infty} \frac{1}{j^{d_k}} \sum_{\ell=0}^{\infty} \frac{\nu_{2i+1}^{i+\ell+1}(d_k)_{2\ell+2i+2}}{2^{2\ell}(2\ell+2i+2)! j^{2\ell+2i+2}} \right\}.
 \end{aligned}$$

Observe that

$$\begin{aligned}
 \sum_{j=1}^{\infty} \frac{1}{j^{d_k}} \sum_{q=1}^{\infty} \frac{(d_k)_{2q}}{(2q)!(2j)^{2q}} &= \sum_{j=1}^{\infty} \frac{1}{j^{d_k}} \sum_{q=0}^{\infty} \frac{(d_k)_{2q}}{(2q)!(2j)^{2q}} - \zeta(d_k) \\
 &= \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{j^{d_k}} \left(\left(1 + \frac{1}{2j}\right)^{-d_k} + \left(1 - \frac{1}{2j}\right)^{-d_k} \right) - \zeta(d_k) \\
 (4.9) \quad &= 2^{d_k} \zeta(d_k) - 2\zeta(d_k) - 2^{d_k-1}.
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 \sum_{m \in \Delta(n)} \frac{2}{m^{d_k}(4r^2 - 1)} &= \frac{2}{(2n)^{d_k}} \left\{ - \left(\frac{1}{n+1} + \frac{1}{n^2-1} \right) \zeta(d_k) - \frac{2^{d_k-1}}{n^2-1} \right. \\
 &\quad + \left(\frac{n(n+2)}{n+1} - \frac{n^2}{n^2-1} \right) \sum_{\ell=1}^{\infty} \frac{(d_k)_{2\ell} \zeta(2\ell + d_k)}{(2\ell)!(2n)^{2\ell}} \\
 &\quad \left. + \sum_{i=0}^{\infty} \frac{1}{n^{2i+1}} \sum_{j=1}^{\infty} \frac{1}{j^{d_k}} \sum_{\ell=0}^{\infty} \frac{\nu_{2i+1}^{i+\ell+1} (d_k)_{2\ell+2i+2}}{2^{2\ell} (2\ell+2i+2)! j^{2\ell+2i+2}} \right\}.
 \end{aligned}
 \tag{4.10}$$

Combining this with (4.8) gives

$$\begin{aligned}
 \sum_{m \in \Delta(n)} \frac{1}{m^{d_k}} E_n^C(T_m) &\sim - \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{4^\ell B_{2\ell} \Gamma(2\ell+2j+d_k+1)}{(2\ell)! \Gamma(d_k+2j+2) n^{2\ell+2j+d_k+1}} \\
 &\quad - \frac{2^{1-d_k}}{n^{d_k}} \left(\frac{1}{n+1} + \frac{1}{n^2-1} \right) \zeta(d_k) \\
 &\quad + \frac{2^{1-d_k}}{n^{d_k}} \left(\frac{n(n+2)}{n+1} - \frac{n^2}{n^2-1} \right) \sum_{\ell=1}^{\infty} \frac{(d_k)_{2\ell} \zeta(2\ell + d_k)}{(2\ell)!(2n)^{2\ell}} \\
 &\quad + \frac{2^{1-d_k}}{n^{d_k}} \sum_{i=0}^{\infty} \frac{1}{n^{2i+1}} \sum_{j=1}^{\infty} \frac{1}{j^{d_k}} \sum_{\ell=0}^{\infty} \frac{\nu_{2i+1}^{i+\ell+1} (d_k)_{2\ell+2i+2}}{2^{2\ell} (2\ell+2i+2)! j^{2\ell+2i+2}}.
 \end{aligned}
 \tag{4.11}$$

Since

$$\frac{1}{n+1} + \frac{1}{n^2-1} = \sum_{j=0}^{\infty} \frac{1}{n^{2j+1}}, \quad \frac{n(n+2)}{n+1} - \frac{n^2}{n^2-1} = n - \sum_{j=0}^{\infty} \frac{1}{n^{2j+1}}.$$

Thus, we can deduce that the asymptotic series on the right hand side of (4.11) consists of negative powers of n with exponents $\{d_k + 2j + 1\}_{j=0}^{\infty}$ and $k \geq 0$. This completes the proof. \square

Corollary 4.2. If the Chebyshev coefficients of $f(x)$ satisfy

$$a_n \sim \sum_{k=0}^{\infty} \frac{\mu_k}{n^{d_k}} + (-1)^n \sum_{k=0}^{\infty} \frac{\gamma_k}{n^{\zeta_k}}, \quad n \rightarrow \infty,
 \tag{4.12}$$

where μ_k, γ_k are constants independent of n and $\{d_k\}_{k=0}^{\infty}$ and $\{\zeta_k\}_{k=0}^{\infty}$ are positive and strictly increasing sequences. Then, we have

$$E_n^C[f] \sim \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\varsigma_{j,k}}{n^{\xi_k+2j+1}}, \quad n \rightarrow \infty,
 \tag{4.13}$$

where $\varsigma_{j,k}$ are constants independent of n and $\{\xi_k\}_{k=0}^{\infty}$ is a strictly increasing sequence and $\{\xi_k\}_{k=0}^{\infty} = \{d_k\}_{k=0}^{\infty} \cup \{\zeta_k\}_{k=0}^{\infty}$.

Proof. It follows from Theorem 4.1. \square

Remark 4.3. A direct consequence of Theorem 4.1 is that the rate of convergence of Clenshaw-Curtis quadrature is $\mathcal{O}(n^{-d_0-1})$. For example, if $f \in X^s$ which implies that $d_0 = s+1$. In this case, we can deduce immediately that the rate of convergence of Clenshaw-Curtis quadrature is $\mathcal{O}(n^{-s-2})$.

For functions $f(x) = (1-x)^\alpha(1+x)^\beta g(x)$ with $\alpha, \beta \geq 0$ are not integers simultaneously and $g(x) \in C^\infty[-1, 1]$, from Theorem 3.2 we know that their Chebyshev coefficients admit the asymptotic of the form (4.2) or (4.3) if β or α is a nonnegative integer. If both α and β are not nonnegative integers, then their Chebyshev coefficients admit the asymptotic of the form (4.12). Similarly, for functions with algebraic-logarithmic singularities of the form (3.16) with α a positive integer. If β is a nonnegative integer, then from (3.24) we see that the asymptotic of their Chebyshev coefficients satisfies the form (4.2). If β is not a nonnegative integer, then the asymptotic of their Chebyshev coefficients satisfies the form (4.12). Therefore, for these cases we mentioned, the error of the Clenshaw-Curtis quadrature always has the asymptotic expansion of the form (4.4) or (4.13).

The error of the form (4.4) or (4.13) is especially suitable for using some convergence acceleration techniques such as Richardson extrapolation and ϵ -algorithm to accelerate the convergence rate of Clenshaw-Curtis quadrature. In particular, the previous function evaluations can be reused in the process of convergence acceleration when n is doubled. In the following we only consider the form (4.4) since the form (4.13) can be dealt with in a similar way. In Algorithm 1 we outline the main steps of the convergence acceleration of Clenshaw-Curtis quadrature by using Richardson extrapolation:

Algorithm 1 Richardson extrapolation for Clenshaw-Curtis quadrature

```

1: Input parameters  $n$  and  $q$ 
2: for  $k = 0 : q$  do
3:   Compute  $R(0, 2^k n) = I_{2^k n}^C[f]$  by FFT;
4: end for
5: for  $j = 0 : q - 1$  do
6:   for  $k = 0 : q - 1 - j$  do
7:     Evaluate  $R(j + 1, 2^k n) = \frac{2^{d_j+1}R(j, 2^{k+1}n) - R(j, 2^k n)}{2^{d_j+1} - 1}$ ;
8:   end for
9: end for
10: Return  $R(q, n)$ .
```

The term $R(q, n)$ achieves a higher order of convergence. More precisely, from the standard theory of Richardson extrapolation we have the following estimate

$$(4.14) \quad I[f] - R(q, n) = \mathcal{O}(n^{-d_q-1}).$$

Note that the Richardson extrapolation scheme $R(q, n)$ reduces to Clenshaw-Curtis quadrature when $q = 0$.

Corollary 4.4. When using the Algorithm 1, the sequence $\{d_k\}_{k=0}^\infty$ can be defined as follows: For functions $f(x) = (1-x)^\alpha(1+x)^\beta g(x)$ with $\alpha, \beta \geq 0$ are not integers simultaneously and $g(x) \in C^\infty[-1, 1]$, we can define

$$(4.15) \quad \{d_k\}_{k=0}^\infty = \begin{cases} \{2\alpha + 2j + 1\}_{j=0}^\infty \cup \{2\beta + 2j + 1\}_{j=0}^\infty, & \text{if } \alpha, \beta \text{ are not integers,} \\ \{2\alpha + 2j + 1\}_{j=0}^\infty, & \text{if } \beta \text{ is an integer,} \\ \{2\beta + 2j + 1\}_{j=0}^\infty, & \text{if } \alpha \text{ is an integer.} \end{cases}$$

For functions $f(x) = (1-x)^\alpha(1+x)^\beta \log(1-x)g(x)$ where α is a positive integer, $\beta \geq 0$ and $g(x) \in C^\infty[-1, 1]$, then we can define

$$(4.16) \quad \{d_k\}_{k=0}^\infty = \begin{cases} \{2\alpha + 2j + 1\}_{j=0}^\infty, & \text{if } \beta \text{ is an integer,} \\ \{2\alpha + 2j + 1\}_{j=0}^\infty \cup \{2\beta + 2j + 1\}_{j=0}^\infty, & \text{otherwise.} \end{cases}$$

Example 4.5. Consider $f(x) = (1-x)^\alpha g(x)$ and $\alpha > 0$ is not an integer. From (3.14) we know that $f \in X^s$ and $s = 2\alpha$. On the other hand, from Corollary 4.4 we see immediately that $d_j = 2j + 2\alpha + 1$ for $j \geq 0$. Thus, the convergence rate of the Richardson extrapolation scheme $R(q, n)$ is

$$(4.17) \quad I[f] - R(q, n) = \mathcal{O}(n^{-2q-s-2}), \quad q \geq 0.$$

This higher order convergence rate is confirmed by numerical experiments in the next section.

Remark 4.6. If $f(x)$ has an interior singularity inside the interval $[-1, 1]$. For example, suppose that

$$(4.18) \quad f(x) = (1-x)^\alpha(1+x)^\beta |x-x_0|^\delta g(x),$$

where $x_0 \in (-1, 1)$ and $\delta \geq 0$ is not an integer. Then, we can first divide the interval $[-1, 1]$ into two parts at $x = x_0$ and then apply Clenshaw-Curtis quadrature or its extrapolation acceleration scheme to the resulting two integrals.

5. NUMERICAL EXPERIMENTS

In this section we present some concrete examples to show the convergence rates of Clenshaw-Curtis quadrature and Richardson extrapolation approach for functions with endpoint singularities. We apply “Acceleration one” and “Acceleration two” to indicate $R(1, n)$ and $R(2, n)$, respectively. For comparison, we also add the rate of convergence of Gauss-Legendre quadrature to the following examples.

Example 5.1. Consider the following function

$$(5.1) \quad f(x) = (1-x)^\alpha(1+x)^\beta e^x,$$

where $\alpha, \beta \geq 0$ are not integers simultaneously. Obviously, Theorem 3.2 implies that $f(x) \in X^s$ where s is defined as in (3.14). From Remark 4.3 and Corollary 4.4 we know that the rate of convergence of Clenshaw-Curtis quadrature is $\mathcal{O}(n^{-d_0-1})$ where $d_0 = s + 1$, while the rate of convergence of the Richardson extrapolation scheme $R(q, n)$ is $I[f] - R(q, n) = \mathcal{O}(n^{-d_q-1})$ and d_q is defined as in (4.15). Numerical results are illustrated in Figure 1 with two different choices of α and β . The left graph of Figure 1 demonstrates the case $\alpha = \frac{1}{2}$ and $\beta = 0$ which implies $d_j = 2j + 2$ for $j \geq 0$. The right graph of Figure 1 demonstrates the case $\alpha = \frac{3}{4}$ and $\beta = \frac{1}{4}$. From (4.15) we can deduce that $d_j = j + \frac{3}{2}$ for $j \geq 0$. It can be observed clearly from Figure 1 that the rate of convergence of Clenshaw-Curtis quadrature is $\mathcal{O}(n^{-s-2})$ and the rate of convergence of the extrapolation scheme $R(q, n)$ is $\mathcal{O}(n^{-d_q-1})$ for $q = 1, 2$, which coincides with our analysis.

Example 5.2. Consider the function

$$(5.2) \quad f(x) = (1-x)^\alpha(1+x)^\beta \log(1-x) \cos(t+1),$$

where α is a positive integer and $\beta \geq 0$. Clearly, $f \in X^s$ and s is defined as in Remark 3.12. In Figure 2 we demonstrate the convergence rate of Clenshaw-Curtis

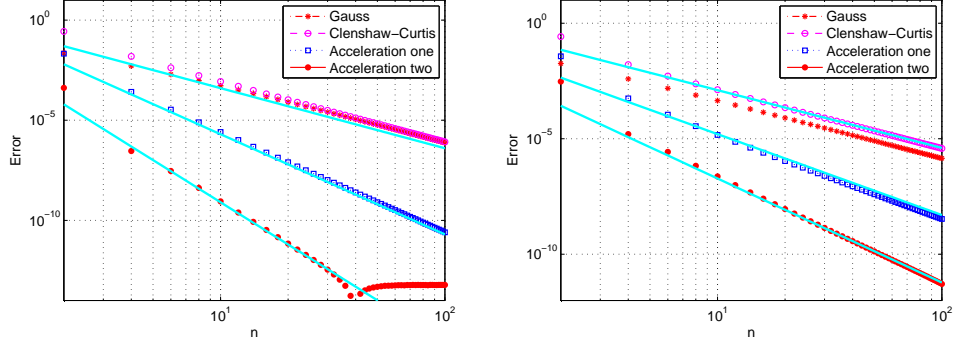


FIGURE 1. Convergence rates of $(n+1)$ -point Clenshaw-Curtis and Gauss quadrature rules for $f(x) = (1-x)^\alpha(1+x)^\beta e^x$ with $\alpha = \frac{1}{2}, \beta = 0$ (left) and $\alpha = \frac{3}{4}, \beta = \frac{1}{4}$ (right). These lines denote $\mathcal{O}(n^{-d_q-1})$ for $q = 0$ (upper), $q = 1$ (middle) and $q = 2$ (lower), and d_q is defined as in (4.15).

and Gauss-Legendre quadrature rules and the Richardson extrapolation schemes $R(1, n)$ and $R(2, n)$. The left graph of Figure 2 demonstrates the case $\alpha = 1$ and $\beta = 0$. In this case, we have from Definition 3.12 and Corollary 4.4 that $s = 2$ and $d_j = 2j + 2\alpha + 1$ for $j \geq 0$. The right graph of Figure 2 demonstrates the case $\alpha = 1$ and $\beta = \frac{1}{2}$. In this case, we have from (4.16) that $s = 1$ and $d_j = j + 2$ for $j \geq 0$. The numerical results shown in Figure 2 are consistent with our theoretical results.

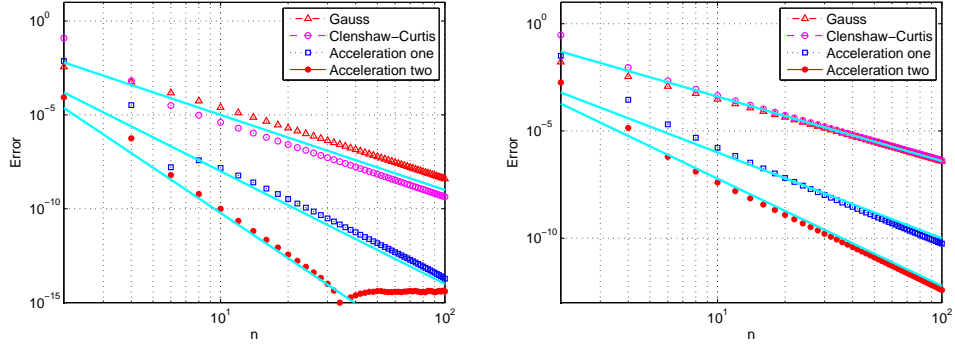


FIGURE 2. Convergence rates of $(n+1)$ -point Clenshaw-Curtis and Gauss quadrature rules for $f(x) = (1-x)^\alpha(1+x)^\beta \log(1-x) \cos(t+1)$ with $\alpha = 1, \beta = 0$ (left) and $\alpha = 1, \beta = \frac{1}{2}$ (right). The line denotes $\mathcal{O}(n^{-d_q-1})$ with $q = 0$ (upper), $q = 1$ (middle) and $q = 2$ (lower).

Example 5.3. Finally, consider the function

$$(5.3) \quad f(x) = \arccos(x^{2m}),$$

where m is a positive integer. Using repeated integration by parts, we obtain the asymptotic of its Chebyshev coefficients

$$a_{2n} \sim \sum_{j=0}^{\infty} \frac{\mu_j}{n^{d_j}},$$

where $d_j = 2j + 2$ for $j \geq 0$ and μ_j are constants depend on m . Here we give explicit expressions for the first three coefficients of μ_j

$$(5.4) \quad \mu_0 = -\frac{\sqrt{2m}}{\pi}, \quad \mu_1 = -\frac{\sqrt{2m}}{4\pi} \left(m - \frac{1}{2}\right), \quad \mu_2 = -\frac{\sqrt{2m}}{16\pi} \left(m^2 - 5m + \frac{9}{4}\right).$$

Moreover, $a_{2n+1} = 0$ for $n \geq 0$ since the function $f(x)$ is even. Obviously, $f \in X^1$ and it satisfies the condition of the Theorem 2.2. Thus, the convergence rate of the $(n+1)$ -point Clenshaw-Curtis quadrature is $\mathcal{O}(n^{-3})$. Figure 3 shows the convergence rates of the Clenshaw-Curtis and Gauss-Legendre quadrature rules and the Richardson extrapolation schemes $R(q, n)$ for the function (5.3) with two different values of m . Clearly, we can see that the convergence rates of both quadrature rules are $\mathcal{O}(n^{-3})$. Moreover, the convergence rate of $R(q, n)$ is $\mathcal{O}(n^{-d_q-1})$.

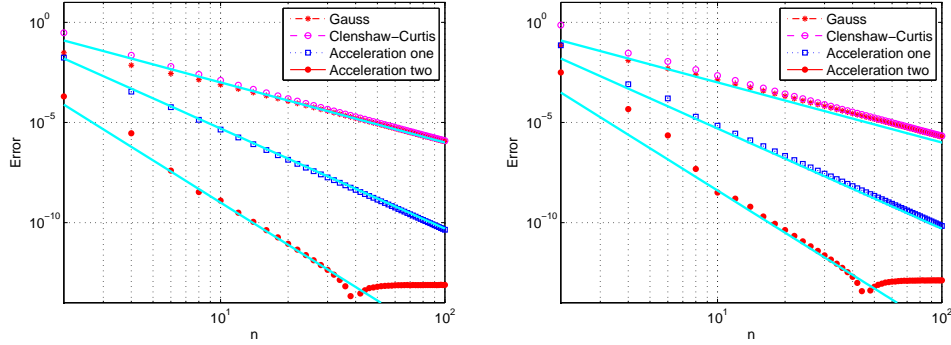


FIGURE 3. Convergence rates of $(n+1)$ -point Clenshaw-Curtis and Gauss quadrature rules and the extrapolation schemes $R(q, n)$ for $f(x) = \arccos(x^{2m})$ with $m = 1$ (left) and $m = 3$ (right). These lines denote $\mathcal{O}(n^{-d_q-1})$ for $q = 0$ (upper), $q = 1$ (middle) and $q = 2$ (lower).

Remark 5.4. From these examples, we can observe that the rate of convergence of Gauss quadrature is almost indistinguishable with that of Clenshaw-Curtis quadrature for functions with endpoint singularities for large n .

6. CONCLUSION

In this paper, we have analyzed the rate of convergence of Clenshaw-Curtis quadrature for functions in X^s which have algebraic or algebraic-logarithmic endpoint singularities. For such functions, we show that the rate of convergence can be further improved to $\mathcal{O}(n^{-s-2})$, which is one power of n better than the optimal estimate given in [22]. Furthermore, an asymptotic error expansion for Clenshaw-Curtis quadrature was obtained, based on which extrapolation schemes such as

Richardson extrapolation was applied to accelerate the convergence of Clenshaw-Curtis quadrature. In contrast to Gauss-Legendre quadrature, Clenshaw-Curtis quadrature is a more powerful scheme to integrate functions with endpoint singularities since its nodes are nested and its quadrature weights can be evaluated efficiently by the inverse Fourier transform.

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